

# Galileons and Naked Singularities

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## Abstract

A simple trace-coupled Galileon model is shown to admit spherically symmetric static solutions with naked spacetime curvature singularities.

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Galileon theories are a class of models for hypothetical scalar fields whose Lagrangians involve multilinear terms of first and second derivatives, but whose nonlinear field equations are still only second order. They may be important for the description of large-scale features in astrophysics as well as for elementary particle theory [4, 7]. Hierarchies of Galileon Lagrangians were first discussed mathematically in [8]. The simplest example involves a single scalar field,  $\phi$ . This Galileon field is usually coupled to all *other* matter through the trace of the energy-momentum tensor,  $\Theta^{(\text{matter})}$ , and is thus gravitation-like by virtue of the similarity between this universal coupling and that of the metric  $g_{\mu\nu}$  to  $\Theta_{\mu\nu}^{(\text{matter})}$  in general relativity. In fact, some Galileon models have been obtained from limits of higher dimensional gravitation theories [6].

In [3] the effects of coupling a Galileon to its own energy-momentum trace were considered, in the flat spacetime limit. Here, general relativistic effects are taken into consideration and additional features of this same model are explored in curved spacetime [5]. While this investigation was in progress, I learned of other work [1] for a related class of models. However, one of the main points of that other study is at variance with the results to follow, namely, the model discussed here admits solutions with *naked singularities* when the energy in the scalar field is finite and not too large, and for which the effective mass of the system is positive. Thus for the simple model at hand there is an open set of *physically acceptable* scalar field data for which curvature singularities are *not* hidden inside event horizons [10, 11]. This would seem to have important implications for the cosmic censorship conjecture [9, 12, 15]. It is worthwhile to note that, in general, naked singularities have observable consequences that differ from those due to black holes [14].

The scalar field part of the action in curved space is

$$A = \frac{1}{2} \int g^{\alpha\beta} \phi_{\alpha} \phi_{\beta} \left( 1 - \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \phi_{\nu}) - \frac{1}{2} g^{\mu\nu} \phi_{\mu} \phi_{\nu} \right) \sqrt{-g} d^4x . \quad (1)$$

This gives a symmetric energy-momentum tensor  $\Theta_{\alpha\beta}$  for  $\phi$  upon variation of the metric.

$$\delta A = \frac{1}{2} \int \sqrt{-g} \Theta_{\alpha\beta} \delta g^{\alpha\beta} d^4x , \quad (2)$$

$$\begin{aligned} \Theta_{\alpha\beta} = & \phi_{\alpha} \phi_{\beta} (1 - g^{\mu\nu} \phi_{\mu} \phi_{\nu}) - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \phi_{\mu} \phi_{\nu} (1 - \frac{1}{2} g^{\rho\sigma} \phi_{\rho} \phi_{\sigma}) \\ & - \phi_{\alpha} \phi_{\beta} \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \phi_{\nu}) + \frac{1}{2} \partial_{\alpha} (g^{\mu\nu} \phi_{\mu} \phi_{\nu}) \phi_{\beta} + \frac{1}{2} \partial_{\beta} (g^{\mu\nu} \phi_{\mu} \phi_{\nu}) \phi_{\alpha} - \frac{1}{2} g_{\alpha\beta} \partial_{\rho} (g^{\mu\nu} \phi_{\mu} \phi_{\nu}) g^{\rho\sigma} \phi_{\sigma} . \end{aligned} \quad (3)$$

It also gives the field equation for  $\phi$  upon variation of the scalar field,  $\mathcal{E}[\phi] = 0$ , where

$$\delta A = - \int \sqrt{-g} \mathcal{E}[\phi] \delta\phi d^4x , \quad (4)$$

$$\mathcal{E}[\phi] = \partial_{\alpha} \left[ g^{\alpha\beta} \phi_{\beta} \sqrt{-g} - g^{\alpha\beta} \phi_{\beta} g^{\mu\nu} \phi_{\mu} \phi_{\nu} \sqrt{-g} - g^{\alpha\beta} \phi_{\beta} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \phi_{\nu}) + \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_{\beta} (g^{\mu\nu} \phi_{\mu} \phi_{\nu}) \right] . \quad (5)$$

Since  $\mathcal{E}[\phi]$  is a total divergence, it easily admits a first integral for static, spherically symmetric configurations. Consider *only* those situations in the following.

For such configurations the metric in generalized Schwarzschild coordinates is [13]

$$(ds)^2 = e^{N(r)} (dt)^2 - e^{L(r)} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2 \theta (d\varphi)^2 . \quad (6)$$

Thus for static, spherically symmetric  $\phi$ , with covariantly conserved energy-momentum tensor (3), Einstein's equations reduce to just a pair of coupled 1st-order nonlinear equations:

$$r^2 \Theta_t^t = e^{-L} (rL' - 1) + 1 , \quad (7)$$

$$r^2 \Theta_r^r = e^{-L} (-rN' - 1) + 1 . \quad (8)$$

These are to be combined with the first integral of the  $\phi$  field equation in this situation. Defining

$$\eta(r) \equiv e^{-L(r)/2} , \quad \varpi(r) \equiv \eta(r) \phi'(r) , \quad (9)$$

that first integral becomes

$$\frac{C e^{-N/2}}{r^2} = \varpi(1 + \varpi^2) + \frac{1}{2} \left( N' + \frac{4}{r} \right) \eta \varpi^2 , \quad (10)$$

where for asymptotically flat spacetime the constant  $C$  is given by  $\lim_{r \rightarrow \infty} r^2 \phi'(r) = C$ . Then upon using

$$\Theta_t^t = \Theta_\theta^\theta = \Theta_\varphi^\varphi = \frac{1}{2} \varpi^2 (1 + \frac{1}{2} \varpi^2) - \eta \varpi^2 \varpi' , \quad (11)$$

$$\Theta_r^r = -\frac{1}{2} \varpi^2 (1 + \frac{3}{2} \varpi^2) - \frac{1}{2} \eta \varpi^3 (N' + \frac{4}{r}) , \quad (12)$$

the remaining steps to follow are clear.

First, for  $C \neq 0$ , one can eliminate  $N'$  from (8) and (10) to obtain an exact expression for  $N$  in terms of  $\eta$ ,  $\varpi$ , and  $C$ :

$$e^{N/2} = \frac{8C}{r\varpi} \frac{\eta - \frac{1}{2} r \varpi^3}{(4\varpi - 2r^2 \varpi^3 - r^2 \varpi^5 + 8r\eta + 12\varpi\eta^2 + 8r\varpi^2\eta)} . \quad (13)$$

If the numerator of this last expression vanishes there is an *event horizon*, otherwise not. When  $\eta = \frac{1}{2} r \varpi^3$  the denominator of (13) is positive definite.

Next, in addition to (7) one can now eliminate  $N$  from either (8) or (10) to obtain two coupled first-order nonlinear equations for  $\eta$  and  $\varpi$ . These can be integrated, at least numerically. Or they can be used to determine analytically the large and small  $r$  behaviors, hence to see if the energy and curvature are finite. For example, again for asymptotically flat spacetime, it follows that

$$e^{L/2} \underset{r \rightarrow \infty}{\sim} 1 + \frac{M}{r} + \frac{1}{4} (6M^2 - C^2) \frac{1}{r^2} + \frac{1}{2} M (5M^2 - 2C^2) \frac{1}{r^3} + O\left(\frac{1}{r^4}\right) , \quad (14)$$

$$e^{N/2} \underset{r \rightarrow \infty}{\sim} 1 - \frac{M}{r} - \frac{1}{2} M^2 \frac{1}{r^2} + \frac{1}{12} M (C^2 - 6M^2) \frac{1}{r^3} + O\left(\frac{1}{r^4}\right) , \quad (15)$$

$$\varpi \underset{r \rightarrow \infty}{\sim} \frac{C}{r^2} \left( 1 + \frac{M}{r} + \frac{3}{2} M^2 \frac{1}{r^2} \right) + O\left(\frac{1}{r^5}\right) , \quad (16)$$

for constant  $C$  and  $M$ .

As of this writing the details of the two remaining first-order ordinary differential equations are not pretty, but the equations are numerically tractable. In terms of the variables defined in (9), Einstein's equation (7) becomes

$$I(r, \varpi, \eta) r \frac{d}{dr} \varpi + J(r, \varpi, \eta) r \frac{d}{dr} \eta = K(r, \varpi, \eta) , \quad (17)$$

$$I(r, \varpi, \eta) = r\eta\varpi^2 , \quad J(r, \varpi, \eta) = -2\eta , \quad (18)$$

$$K(r, \varpi, \eta) = \frac{1}{2} r^2 \varpi^2 (1 + \frac{1}{2} \varpi^2) + \eta^2 - 1 . \quad (19)$$

But worse than that, in light of (13) Einstein's equation (8) becomes

$$F(r, \varpi, \eta) r \frac{d}{dr} \varpi + G(r, \varpi, \eta) r \frac{d}{dr} \eta = H(r, \varpi, \eta) , \quad (20)$$

$$F(r, \varpi, \eta) = -4\eta [2r^3\varpi^6 + 3r^3\varpi^8 + 16\varpi\eta + 4r\varpi^4 + 16r\eta^2 + 48\varpi\eta^3 + 48r\varpi^2\eta^2 + 12r\varpi^4\eta^2 - 12r^2\varpi^5\eta] , \quad (21)$$

$$G(r, \varpi, \eta) = 8\eta\varpi^2 [2r^2\varpi^2 + 3r^2\varpi^4 - 12\eta^2 + 12r\varpi^3\eta + 4] , \quad (22)$$

$$H(r, \varpi, \eta) = \varpi [8\eta\varpi (4r\varpi^3 - 4\eta + 2r^2\varpi^2\eta + 3r^2\varpi^4\eta + 12r\varpi^3\eta^2 - 12\eta^3) + (4 + 3r^2\varpi^4 + 2r^2\varpi^2 + 12\eta^2) (4\varpi - r^2\varpi^5 - 2r^2\varpi^3 + 8r\varpi^2\eta + 8r\eta + 12\varpi\eta^2)] . \quad (23)$$

As a representative example with  $\varpi > 0$ , (17) and (20) were integrated numerically to obtain the results shown in Figure 1, for data initialized as  $\varpi|_{r=1} = 0.5$  and  $\eta|_{r=1} = 1$ . Evidently it is true that  $\eta(r) \neq \frac{1}{2}r\varpi^3(r)$  for this case, so  $e^{N(r)}$  does not vanish for any  $r > 0$  and there is no event horizon.

However, there is a geometric singularity at  $r = 0$  with divergent scalar curvature:  $\lim_{r \rightarrow 0} r^{3/2}R = \text{const.}$  Since  $R = -\Theta_\mu^\mu$ , and  $\lim_{r \rightarrow 0} \varpi$  is finite, this divergence in  $R$  comes from the last term in (12), which in turn comes from the second term in  $A$ , i.e. the covariant  $\partial\phi\partial\phi\partial^2\phi$  in (1). In fact, it is not difficult to establish analytically for a class of solutions of the model, for which the example in Figure 1 is representative, the following limiting behavior holds.

$$\lim_{r \rightarrow 0} (e^{L/2}/\sqrt{r}) = \ell , \quad \lim_{r \rightarrow 0} (\sqrt{r}e^{N/2}) = n , \quad \lim_{r \rightarrow 0} \varpi = p , \quad \lim_{r \rightarrow 0} (\phi'/\sqrt{r}) = p\ell , \quad (24)$$

where  $\ell$ ,  $n$ , and  $p$  are constants related to the constant  $C$  in (10):

$$2C = 3np^2/\ell . \quad (25)$$

It follows that for solutions in this class,

$$\lim_{r \rightarrow 0} r^{3/2}R = pC/n . \quad (26)$$

For the example shown in Figure 1:  $\ell \approx 1.5$ ,  $n \approx 0.086$ ,  $p \approx 3.3$ ,  $C \approx 0.94$ , and  $pC/n \approx 36$ .

For the same  $\eta|_{r=1} = 1$ , further numerical results show there are also curvature singularities without horizons for smaller  $\varpi|_{r=1} > 0$ , but event horizons are present for larger scalar fields (roughly when  $\varpi|_{r=1} > 2/3$ ). A more precise and complete characterization of the data set  $\{\varpi|_{r=1}, \eta|_{r=1}\}$  for which there are naked singularities is in progress, but it is already evident from the preceding remarks that the set has nonzero measure.

The energy contained in *only* the scalar field in the curved spacetime is given by

$$E_{\text{Galileon}} = \int_0^\infty \mathcal{H}(r) dr = \int_{-\infty}^\infty e^s \mathcal{H}(e^s) ds , \quad (27)$$

$$\mathcal{H}(r) \equiv 4\pi r^2 e^{L/2} e^{N/2} \Theta_t^t = 2\pi e^{2s} e^{L/2} e^{N/2} \varpi^2(s) (1 + \frac{1}{2}\varpi^2(s)) - 4\pi e^s e^{N/2} \varpi^2(s) \frac{d}{ds} \varpi(s) . \quad (28)$$

For the above numerical example, the integrand  $e^s \mathcal{H}(e^s)$  is shown in Figure 2. Evidently,  $E_{\text{Galileon}}$  is finite in this case. It is also clear from the Figures that the Galileon field has significant effects on the geometry in the vicinity of the peak of its radial energy density. There the metric coefficients are greatly distorted from the familiar Schwarzschild values, and as a consequence, the horizon is eliminated.

It remains to investigate the stability of the static solutions described above, and to consider the dynamical evolution of generic Galileon and other matter field initial data, along the lines of [2], to determine under what physical conditions the naked singularities discussed here are actually formed.

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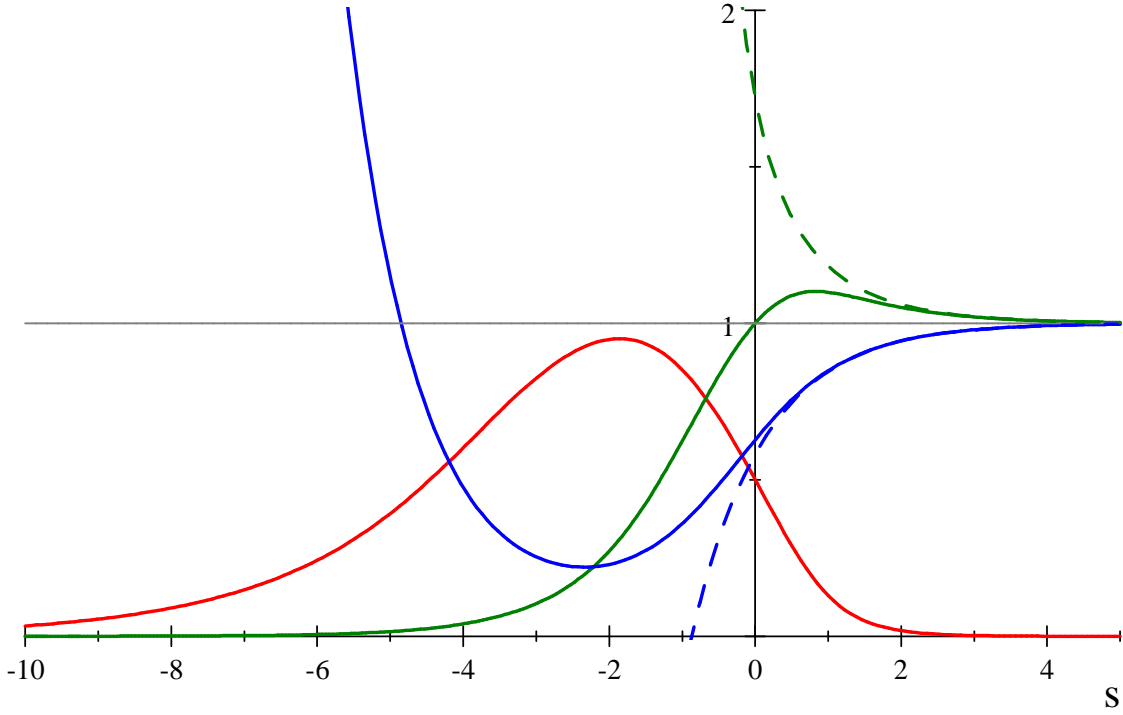


Fig. 1: For initial values  $\varpi(s)|_{s=0} = 0.5$  and  $\eta(s)|_{s=0} = 1.0$ ,  $d\phi/dr = \varpi/\eta$  is shown in red,  $e^L = 1/\eta^2$  in green, and  $e^N$  in blue, where  $r = e^s$ . For comparison, Schwarzschild  $e^L$  and  $e^N$  are also shown as resp. green and blue dashed curves for the same  $M \approx 0.21$  [16].

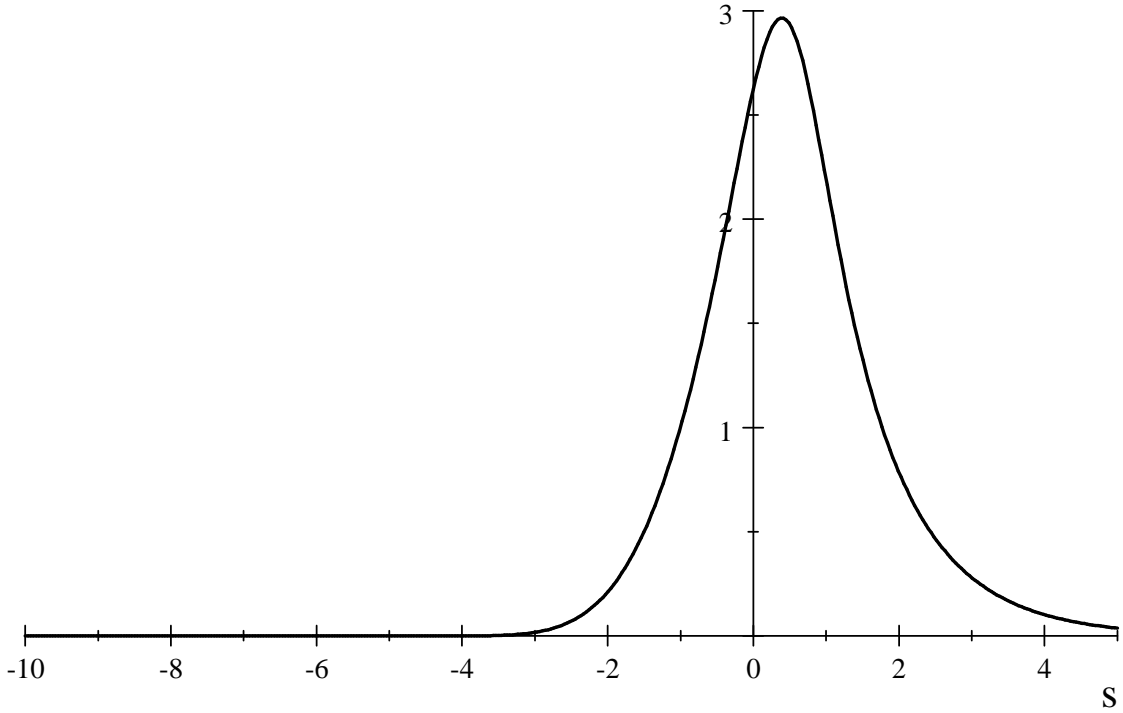


Fig. 2:  $e^s \mathcal{H}(e^s)$  for  $\varpi(s)|_{s=0} = 0.5$  and  $\eta(s)|_{s=0} = 1.0$ , where  $r = e^s$ .

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- [1] C. Charmousis, B. Goutéraux, and E. Kiritsis, “Higher-derivative scalar-vector-tensor theories: black holes, Galileons, singularity cloaking and holography” arXiv:1206.1499 [hep-th]
  - [2] M. W. Choptuik, “Universality and Scaling in Gravitational Collapse of a Massless Scalar Field” Phys. Rev. Lett 70 (1993) 9-12
  - [3] T. Curtright and D. Fairlie, “Geons of Galileons” arXiv:1206.3616 [hep-th]
  - [4] C. de Rham, “Galileons in the Sky” arXiv:1204.5492 [astro-ph.CO]
  - [5] C. Deffayet, S. Deser, and G. Esposito-Farese, “Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress tensors” Phys. Rev. D80 (2009) 064015, arxiv:0906.1967 [hep-th]; C. Deffayet, G. Esposito-Farese, and A. Vikman, “Covariant Galileon” Phys. Rev. D79 (2009) 084003, arXiv:0901.1314 [hep-th]
  - [6] G. R. Dvali, G. Gabadadze, and M. Porrati, “4-D gravity on a brane in 5-D Minkowski space” Phys. Lett. B 485 (2000) 208 [arXiv:hep-th/0005016]; M. A. Luty, M. Porrati and R. Rattazzi, “Strong interactions and stability in the DGP model” JHEP 0309 (2003) 029 [arXiv:hep-th/0303116]; A. Nicolis and R. Rattazzi, “Classical and quantum consistency of the DGP model” JHEP 0406 (2004) 059 [arXiv:hep-th/0404159]
  - [7] D. Fairlie, “Comments on Galileons” J. Phys. A44 (2011) 305201, arXiv:1102.1594 [hep-th]
  - [8] D. B. Fairlie and J. Govaerts, “Euler hierarchies and universal equations” J. Math. Phys. 33 (1992) 3543-3566 [arXiv hep-th/9204074]; D. B. Fairlie and J. Govaerts, “Linearisation of Universal Field Equations” J. Phys. A26 (1993) 3339-3347 [arXiv:hep-th/9212005]; D. B. Fairlie, J. Govaerts, and A. Morozov “Universal field equations with covariant solutions” Nucl. Phys. B373 (1992) 214-232 [arXiv hep-th/9110022]
  - [9] R. Penrose, “The Question of Cosmic Censorship” J. Astrophys. Astr. 20 (1999) 233–248
  - [10] Horizon-less solutions with naked singularities have also appeared in recent studies on solutions to higher dimensional Einstein equations in vacuum, suitable for describing intersecting brane solutions in string/M theory. In particular, see S. K. Rama, “Static brane-like vacuum solutions in  $D \geq 5$  dimensional spacetime with positive ADM mass but no horizon” arXiv:1111.1897 [hep-th]
  - [11] In the context of a different Galileon model, numerical evidence of naked singularities is also mentioned in passing by M. Rinaldi, “Galileon Black Holes” arXiv:1208.0103v1 [gr-qc]
  - [12] T. P. Singh, “Gravitational Collapse, Black Holes and Naked Singularities” J. Astrophys. Astr. 20 (1999) 221–232 [arXiv:gr-qc/9805066]
  - [13] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology*, Dover Publications (1987)
  - [14] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, “Role of the scalar field in gravitational lensing” Astron. Astrophys. 337 (1998) 1-8; K. S. Virbhadra and G. F. R. Ellis, “Gravitational lensing by naked singularities” Phys. Rev. D 65 (2002) 103004; K. S. Virbhadra and C. R. Keeton, “Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities” Phys. Rev. D 77 (2008) 124014, arXiv:0710.2333
  - [15] R. Wald, “Gravitational Collapse and Cosmic Censorship” [arXiv:gr-qc/9710068]
  - [16] Figure 1 in the original version of this manuscript displayed the Schwarzschild metric for an erroneous mass (namely,  $M \approx 0.36$ , rather than the correct value that is used here,  $M \approx 0.21$ ).